On chromatic number of colored mixed graphs

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Abstract

An (m,n)-colored mixed graph G is a graph with its arcs having one of the m different colors and edges having one of the n different colors. A homomorphism f of an (m,n)-colored mixed graph G to an (m,n)-colored mixed graph H is a vertex mapping such that if uv is an arc (edge) of color c in G, then $f(u)f(v)$ is an arc (edge) of color c in H. The *(m,n)-colored mixed chromatic number* $\chi_{(m,n)}(G)$ of an (m,n)-colored mixed graph G is the order (number of vertices) of the smallest homomorphic image of G. This notion was introduced by Nešetřil and Raspaud (2000, J. Combin. Theory, Ser. B 80, 147–155). They showed that $\chi_{(m,n)}(G) \leq k(2m+n)^{k-1}$ where G is a k-acyclic colorable graph. We proved the tightness of this bound. We also showed that the acyclic chromatic number of a graph is bounded by $k^2 + k^{2+\lceil log_{(2m+n)} log_{(2m+n)} k \rceil}$ if its (m,n)-colored mixed chromatic number is at most k. Furthermore, using probabilistic method, we showed that for graphs with maximum degree Δ its (m,n)-colored mixed chromatic number is at most $2(\Delta-1)^{2m+n}(2m+n)^{\Delta-1}$. In particular, the last result directly improves the upper bound $2\Delta^2 2^{\Delta}$ of oriented chromatic number of graphs with maximum degree Δ, obtained by Kostochka, Sopena and Zhu (1997, J. Graph Theory 24, 331–340) to $2(\Delta-1)^2 2^{\Delta-1}$. We also show that there exists a graph with maximum degree Δ and (m,n)-colored mixed chromatic number at least $(2m+n)^{\Delta/2}$.

Keywords: colored mixed graphs, acyclic chromatic number, graphs with bounded maximum degree, arboricity, chromatic number.

1 Introduction

An *(m,n)-colored mixed graph* $G = (V, A \cup E)$ is a graph G with set of vertices V, set of arcs A and set of edges E where each arc is colored by one of the m colors $\alpha_1, \alpha_2, ..., \alpha_m$ and each edge is colored by one of the n colors $\beta_1, \beta_2, ..., \beta_n$. We denote the number of vertices and the number of edges of the underlying graph of G by v_G and e_G, respectively. Also, we will consider only those (m,n)-colored mixed graphs for which the underlying undirected graph is simple. Nešetřil and Raspaud [5] generalized the notion of vertex coloring and chromatic number for (m,n)-colored mixed graphs by definining colored homomorphism.

Let $G = (V_1, A_1 \cup E_1)$ and $H = (V_2, A_2 \cup E_2)$ be two (m,n)-colored mixed graphs. A colored homomorphism of G to H is a function $f : V_1 \to V_2$ satisfying

$$uv \in A_1 \Rightarrow f(u)f(v) \in A_2,$$

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$$uv \in E_1 \Rightarrow f(u)f(v) \in E_2,$$

and the color of the arc or edge linking $f(u)$ and $f(v)$ is the same as the color of the arc or the edge linking u and v [5]. We write $G \to H$ whenever there exists a homomorphism of G to H.

Given an (m,n)-colored mixed graph G let H be an (m,n)-colored mixed graph with minimum *order* (number of vertices) such that $G \to H$. Then the order of H is the *(m,n)-colored mixed chromatic number* $\chi_{(m,n)}(G)$ of G. For an undirected simple graph G, the maximum (m,n)-colored mixed chromatic number taken over all (m,n)-colored mixed graphs having underlying undirected simple graph G is denoted by $\chi_{(m,n)}(G)$. Let $\mathcal{F}$ be a family of undirected simple graphs. Then $\chi_{(m,n)}(\mathcal{F})$ is the maximum of $\chi_{(m,n)}(G)$ taken over all $G \in \mathcal{F}$.

Note that a $(0,1)$-colored mixed graph G is nothing but an undirected simple graph while $\chi_{(0,1)}(G)$ is the ordinary chromatic number. Similarly, the study of $\chi_{(1,0)}(G)$ is the study of oriented chromatic number which is considered by several researchers in the last two decades (for details please check the recent updated survey [8]). Alon and Marshall [1] studied the homomorphism of $(0,n)$-colored mixed graphs with a particular focus on $n = 2$.

A simple graph G is *k-acyclic colorable* if we can color its vertices with k colors such that each color class induces an independent set and any two color class induces a forest. The *acyclic chromatic number* $\chi_a(G)$ of a simple graph G is the minimum k such that G is k-acyclic colorable. Nešetřil and Raspaud [5] showed that $\chi_{(m,n)}(G) \leq k(2m+n)^{k-1}$ where G is a k-acyclic colorable graph. As planar graphs are 5-acyclic colorable due to Borodin [2], the same authors implied $\chi_{(m,n)}(\mathcal{P}) \leq 5(2m+n)^4$ for the family $\mathcal{P}$ of planar graphs as a corollary. This result, in particular, implies $\chi_{(1,0)}(\mathcal{P}) \leq 80$ and $\chi_{(0,2)}(\mathcal{P}) \leq 80$ (independently proved before in [7] and [1], respectively).

Let $\mathcal{A}_k$ be the family of graphs with acyclic chromatic number at most k. Ochem [6] showed that the upper bound $\chi_{(1,0)}(\mathcal{A}_k) \leq 80$ is tight. We generalize it for all $(m,n) \neq (0,1)$ to show that the upper bound $\chi_{(m,n)}(\mathcal{A}_k) \leq k(2m+n)^{k-1}$ obtained by Nešetřil and Raspaud [5] is tight. This implies that the upper bound $\chi_{(m,n)}(\mathcal{P}) \leq 5(2m+n)^4$ cannot be improved using the upper bound of $\chi_{(m,n)}(\mathcal{A}_5)$.

The arboricity $arb(G)$ of a graph G is the minimum k such that the edges of G can be decomposed into k forests. Kostochka, Sopena and Zhu [3] showed that given a simple graph G, the acyclic chromatic number $\chi_a(G)$ of G is also bounded by a function of $\chi_{(1,0)}(G)$. We generalize this result for all $(m,n) \neq (0,1)$ by showing that for a graph G with $\chi_{(m,n)}(G) \leq k$ we have $\chi_a(G) \leq k^2 + k^{2+\lceil log_2 log_p k \rceil}$ where $p = 2m+n$. Our bound slightly improves the bound obtained by Kostochka, Sopena and Zhu [3] for $(m,n) = (1,0)$. For achieving this result we first establish some relations among arboricity of a graph, (m,n)-colored mixed chromatic number and acyclic chromatic number.

Let $\mathcal{G}_\Delta$ be the family of graphs with maximum degree Δ. Kostochka, Sopena and Zhu [3] proved that $2^{\Delta/2}\chi_{(1,0)}(\mathcal{G}_\Delta) \leq 2\Delta^2 2^\Delta$. We improve this result in a generalized setting by proving $p^{\Delta/2} \leq \chi_{(m,n)}(\mathcal{G}_\Delta) \leq 2(\Delta-1)^p p^{\Delta-1}$ for all $(m,n) \neq (0,1)$ where $p = 2m+n$.

2 Preliminaries

A *special 2-path* uvw of an (m,n)-colored mixed graph G is a 2-path satisfying one of the following conditions:

(i) uv and vw are edges of different colors,

(ii) uv and vw are arcs (possibly of the same color),

(iii) uv and wv are arcs of different colors,

(iv) vu and vw are arcs of different colors,

(v) exactly one of uv and vw is an edge and the other is an arc.

Observation 1. *The endpoints of a special 2-path must have different image under any homomorphism of G.*

Proof. Let uvw be a special 2-path in an (m,n)-colored mixed graph G. Let $f : G \to H$ be a colored homomorphism of G to an (m,n)-colored mixed graph H such that $f(u) = f(w)$. Then $f(u)f(v)$ and $f(w)f(v)$ will induce parallel edges in the underlying graph of H. But as we are dealing with (m,n)-colored mixed graphs with underlying simple graphs, this is not possible. □

Let $G = (V, A \cup E)$ be an (m,n)-colored mixed graph. Let uv be an arc of G with color α_i for some $i \in \{1, 2, ..., m\}$. Then u is a $-\alpha_i$*-neighbor* of v and v is a $+\alpha_i$*-neighbor* of u. The set of all $+\alpha_i$-neighbors and $-\alpha_i$-neighbors of v is denoted by $N^{+\alpha_i}(v)$ and $N^{-\alpha_i}(v)$, respectively. Similarly, let uv be an edge of G with color β_i for some $i \in \{1, 2, ..., n\}$. Then u is a β_i*-neighbor* of v and the set of all β_i-neighbors of v is denoted by $N^{\beta_i}(v)$. Let $\vec{a} = (a_1, a_2, ..., a_j)$ be a j*-vector* such that $a_i \in \{\pm\alpha_1, \pm\alpha_2, ..., \pm\alpha_m, \beta_1, \beta_2, ..., \beta_n\}$ where $i \in \{1, 2, ..., j\}$. Let $J = (v_1, v_2, ..., v_j)$ be a j*-tuple* (without repetition) of vertices from G. Then we define the set $N^{\vec{a}}(J) = \{v \in V | v \in N^{a_i}(v_i) \textit{ for all } 1 \le i \le j\}$. Finally, we say that G has property $Q^{t,j}_{g(j)}$ if for each j-vector $\vec{a}$ and each j-tuple J we have $|N^{\vec{a}}(J)| \ge g(j)$ where $j \in \{0, 1, ..., t\}$ and $g : \{0, 1, ..., t\} \to \{0, 1, ...\infty\}$ is an integral function.

3 On graphs with bounded acyclic chromatic number

First we will construct examples of (m,n)-colored mixed graphs $H_k^{(m,n)}$ with acyclic chromatic number at most k and $\chi_{(m,n)}(H_k^{(m,n)}) = k(2m+n)^{k-1}$ for all $k \ge 3$ and for all $(m,n) \neq (0,1)$. This, along with the upper bound established by Nešetřil and Raspaud [5], will imply the following result:

Theorem 3.1. *Let $\mathcal{A}_k$ be the family of graphs with acyclic chromatic number at most k. Then $\chi_{(m,n)}(\mathcal{A}_k) = k(2m+n)^{k-1}$ for all $k \ge 3$ and for all $(m,n) \neq (0,1)$.*

Proof. First we will construct an (m,n)-colored mixed graph $H_k^{(m,n)}$, where $p = 2m + n \ge 2$, as follows. Let A_{k-1} be the set of all $(k-1)$-vectors. Thus, $|A_{k-1}| = p^{k-1}$.

Define B_i as a set of $(k-1)$ vertices $B_i = \{b_1^i, b_2^i, ..., b_{k-1}^i\}$ for all $i \in \{1, 2, ..., k\}$ such that $B_r \cap B_s = \emptyset$ when $r \neq s$. The vertices of B_i's are called *bottom* vertices for each $i \in \{1, 2, ..., k\}$. Furthermore, let $TB_i = (b_1^i, b_2^i, ..., b_{k-1}^i)$ be a $(k-1)$-tuple.

After that define the set of vertices $T_i = \{t^i_{\vec{a}} | t^i_{\vec{a}} \in N^{\vec{a}}(TB_i) \textit{ for all } \vec{a} \in A_{k-1}\}$ for all $i \in \{1, 2, ..., k\}$. The vertices of T_i's are called *top* vertices for each $i \in \{1, 2, ..., k\}$. Observe that there are p^{k-1} vertices in T_i for each $i \in \{1, 2, ..., k\}$.

Note that the definition of T_i already implies some colored arcs and edges between the set of vertices B_i and T_i for all $i \in \{1, 2, ..., k\}$.

As $p \ge 2$ it is possible to construct a special 2-path. Now for each pair of vertices $u \in T_i$ and $v \in T_j$ $(i \neq j)$, construct a special 2-path $uw_{uv}v$ and call these new vertices w_{uv} as *internal* vertices for all $i, j \in \{1, 2, ..., k\}$. This so obtained graph is $H_k^{(m,n)}$.

Now we will show that $\chi_{(m,n)}(H_k^{(m,n)}) \geq k(2m+n)^{k-1}$. Let $\vec{a} \neq \vec{a'}$ be two distinct $(k-1)$-vectors. Assume that the j^th co-ordinate of $\vec{a}$ and $\vec{a'}$ is different. Then note that $t^i_{\vec{a}} b^i_j t^i_{\vec{a'}}$ is a special 2-path. Therefore, $t^i_{\vec{a}}$ and $t^i_{\vec{a}}$ must have different homomorphic image under any homomorphism. Thus, all the vertices in T_i must have distinct homomorphic image under any homomorphism. Moreover, as a vertex of T_i is connected by a special 2-path with a vertex of T_j for all $i \neq j$, all the top vertices must have distinct homomorphic image under any homomorphism. It is easy to see that $|T_i| = p^{k-1}$ for all $i \in \{1, 2, ..., k\}$. Hence $\chi_{(m,n)}(H_k^{(m,n)}) \geq \sum_{i=1}^{k} |T_i| = k(2m+n)^{k-1}$.

Then we will show that $\chi_a(H_k^{(m,n)}) \leq k$. From now on, by $H_k^{(m,n)}$, we mean the underlying undirected simple graph of the (m,n)-colored mixed graph $H_k^{(m,n)}$. We will provide an acyclic coloring of this graph with $\{1, 2, ..., k\}$. Color all the vertices of T_i with i for all $i \in \{1, 2, ..., k\}$. Then color all the vertices of B_i with distinct $(k-1)$ colors from the set $\{1, 2, ..., k\} \setminus \{i\}$ of colors for all $i \in \{1, 2, ..., k\}$. Note that each internal vertex have exactly two neighbors. Color each internal vertex with a color different from its neighbors. It is easy to check that this is an acyclic coloring.

Therefore, we showed that $\chi_{(m,n)}(\mathcal{A}_k) \geq k(2m+n)^{k-1}$ while, on the other hand, Nešetřil and Raspaud [5] showed that $\chi_{(m,n)}(\mathcal{A}_k) \leq k(2m+n)^{k-1}$ for all $k \geq 3$ and for all $(m,n) \neq (0,1)$. $\square$

Consider a complete graph K_t. Replace all its edges by a 2-path to obtain the graph S. For all $(m,n) \neq (0,1)$, it is possible to assign colored edges/arcs to the edges of S such that it becomes an (m,n)-colored mixed graph with t vertices that are pairwise connected by a special 2-path. Therefore, by Observation 1 we know that $\chi_{(m,n)}(S) \geq t$ whereas, it is easy to note that S has arboricity 2. Thus, the (m,n)-colored mixed chromatic number is not bounded by any function of arboricity. Though the reverse type of bound exists. Kostochka, Sopena and Zhu [3] proved such a bound for $(m,n) = (1,0)$. We generalize their result for all $(m,n) \neq (0,1)$.

Theorem 3.2. *Let G be an (m,n)-colored mixed graph with $\chi_{(m,n)}(G) = k$ where $p = 2m+n \geq 2$. Then $arb(G) \leq \lceil log_p k + k/2 \rceil$.*

Proof. Let G' be an arbitrary labeled subgraph of G consisting $v_{G'}$ vertices and $e_{G'}$ edges. We know from Nash-Williams' Theorem [4] that the arboricity $arb(G)$ of any graph G is equal to the maximum of $\lceil e_{G'}/(v_{G'}-1) \rceil$ over all subgraphs G' of G. So it is sufficient to prove that for any subgraph G' of G, $e_{G'}/(v_{G'}-1) \leq log_p k + k/2$. As G' is a labeled graph, so there are $p^{e_{G'}}$ different (m,n)-colored mixed graphs with underlying graph G'. As $\chi_{(m,n)}(G) = k$, there exits a homomorphism from G' to a (m,n)-colored mixed graph G_k which has the complete graph on k vertices as its underlying graph. Note that the number of possible homomorphisms of G' to G_k is at most $k^{v_{G'}}$. For each such homomorphism of G' to G_k there are at most $p^{\binom{k}{2}}$ different (m,n)-colored mixed graphs with underlying labeled graph G' as there are $p^{\binom{k}{2}}$ choices of G_k. Therefore,

$$p^{\binom{k}{2}}.k^{v_{G'}} \geq p^{e_{G'}} \tag{1}$$

which implies

$$log_p k \geq (e_{G'}/v_{G'}) - \binom{k}{2}/v_{G'}. \tag{2}$$

If $v_{G'} \leq k$, then $e_{G'}/(v_{G'}-1) \leq v_{G'}/2 \leq k/2$. Now let $v_{G'} > k$. We know that $\chi_{(m,n)}(G') \leq \chi_{(m,n)}(G) = k$. So

$$
\begin{aligned}
log_p k &\geq \frac{e_{G'}}{v_{G'}} - \frac{k(k-1)}{2v_{G'}} \\
&\geq \frac{e_{G'}}{(v_{G'}-1)} - \frac{e_{G'}}{v_{G'}(v_{G'}-1)} - \frac{k-1}{2} \\
&\geq \frac{e_{G'}}{(v_{G'}-1)} - 1/2 - k/2 + 1/2 \\
&\geq \frac{e_{G'}}{(v_{G'}-1)} - k/2.
\end{aligned}
$$

Therefore, $\frac{e_{G'}}{(v_{G'}-1)} \leq log_p k + k/2$. □

We have seen that the (m,n)-colored mixed chromatic number of a graph G is bounded by a function of the acyclic chromatic number of G. Here we show that it is possible to bound the acyclic chromatic number of a graph in terms of its (m,n)-colored mixed chromatic number and arboricity. Our result is a generalization of a similar result proved for $(m,n) = (1,0)$ by Kostochka, Sopena and Zhu [3].

Theorem 3.3. *Let G be an (m,n)-colored mixed graph with $arb(G) = r$ and $\chi_{(m,n)}(G) = k$ where $p = 2m+n \geq 2$. Then $\chi_a(G) \leq k^{\lceil log_p r \rceil + 1}$.*

Proof. First we rename the following symbols: $\alpha_1 = a_0, -\alpha_1 = a_1, \alpha_2 = a_2, -\alpha_2 = a_3, ..., \alpha_m = a_{2m-2}, -\alpha_m = a_{2m-1}, \beta_1 = a_{2m}, \beta_2 = a_{2m+1}, ..., \beta_n = a_{2m+n-1}$.

Let G be a graph with $\chi_{(m,n)}(G) = k$ where $2m+n = p$. Let $v_1, v_2, ..., v_t$ be some ordering of the vertices of G. Now consider the (m,n)-colored mixed graph G_0 with underlying graph G such that for any $i < j$ we have $v_j \in N^{a_0}(v_i)$ whenever v_iv_j is an edge of G.

Note that the edges of G can be covered by r edge disjoint forests $F_1, F_2, ..., F_r$ as $arb(G) = r$. Let s_i be the number i expressed with base p for all $i \in \{1,2,...,r\}$. Note that s_i can have at most $s = \lceil log_p r \rceil$ digits.

Now we will construct a sequence of (m,n)-colored mixed graphs $G_1, G_2, ..., G_s$ each having underlying graph G. For a fixed $l \in \{1,2,...,s\}$ we will describe the construction of G_l. Let $i < j$ and v_iv_j is an edge of G. Suppose v_iv_j is an edge of the forest $F_{l'}$ for some $l' \in \{1,2,...,r\}$. Let the l^{th} digit of $s_{l'}$ be $s_{l'}(l)$. Then G_l is constructed in a way such that we have $v_j \in N^{a_{s_{l'}(l)}}(v_i)$ in G_l.

Note that there is a homomorphism $f_l : G_l \to H_l$ for each $l \in \{1,2,...,s\}$ such that H_l is an (m,n)-colored mixed graph on k vertices. Now we claim that $f(v) = (f_0(v), f_1(v), ..., f_s(v))$ for each $v \in V(G)$ is an acyclic coloring of G.

For adjacent vertices u, v in G clearly we have $f(v) \neq f(u)$ as $f_0(v) \neq f_0(u)$. Let C be a cycle in G. We have to show that at least 3 colors have been used to color this cycle with respect to the coloring given by f. Note that in C there must be two incident edges uv and vw such that they belong to different forests, say, F_i and $F_{i'}$, respectively. Now suppose that C received two colors with respect to f. Then we must have $f(u) = f(w) \neq f(v)$. In particular we must have $f_0(u) = f_0(w) \neq f_0(v)$. To have that we must also have $u, w \in N^{a_i}(v)$ for some $i \in \{0,1,...,p-1\}$ in G_0. Let s_i and $s_{i'}$ differ in their j^{th} digit. Then in G_j we have $u \in N^{a'_i}(v)$ and $w \in N^{a''_i}(v)$ for some $i' \neq i''$. Then we must have $f_j(u) \neq f_j(w)$. Therefore, we also have $f(u) \neq f(w)$. Thus, the cycle C cannot be colored with two colors under the coloring f. So f is indeed an acyclic coloring of G. □

Thus, combining Theorem 3.2 and 3.3 we have $\chi_a(G) \leq k^{\lceil log_p \lceil log_p k + k/2 \rceil \rceil + 1}$ for $\chi_{(m,n)}(G) = k$ where $p = 2m+n \geq 2$. However, we managed to obtain the following better bound.

Theorem 3.4. *Let G be an (m,n)-colored mixed graph with $\chi_{(m,n)}(G) = k \geq 4$ where $p = 2m+n \geq 2$. Then $\chi_a(G) \leq k^2 + k^{2+\lceil log_2 log_p k \rceil}$.*

Proof. Let t be the maximum real number such that there exists a subgraph G' of G with $v_{G'} \geq k^2$ and $e_{G'} \geq t.v_{G'}$. Let G'' be the biggest subgraph of G with $e_{G''} > t.v_{G''}$. Thus, by maximality of t, $v_{G''} < k^2$.

Let $G_0 = G - G''$. Hence $\chi_a(G) \leq \chi_a(G_0) + k^2$. By maximality of G'', for each subgraph H of G_0, we have $e_H \leq t.v_H$.

If $t \leq \frac{v_H - 1}{2}$, then $e_H \leq (t+1/2)(v_H - 1)$. If $t > \frac{v_H-1}{2}$, then $\frac{v_H}{2} < t + 1/2$. So $e_H \leq \frac{(v_H-1).v_H}{2} \leq (t+1/2)(v_H-1)$. Therefore, $e_H \leq (t+1/2)(v_H-1)$ for each subgraph H of G_0.

By Nash-Williams' Theorem [4], there exists $r = \lceil t+1/2 \rceil$ forests $F_1, F_2, \cdots, F_r$ which covers all the edges of G_0. We know from Theorem 3.3 $\chi_a(G_0) \leq k^{s+1}$ where $s = \lceil log_p r \rceil$.

Using inequality (2) we get $log_p k \geq t - 1/2$. Therfore

$$s = \lceil log_p(\lceil t+1/2 \rceil) \rceil \leq \lceil log_p(1 + \lceil log_p k \rceil) \rceil \leq 1 + \lceil log_p log_p k \rceil.$$

Hence $\chi_a(G) \leq k^2 + k^{2+\lceil log_p log_p k \rceil}$. □

Our bound, when restricted to the case of $(m,n) = (1,0)$, slightly improves the existing bound [3].

4 On graphs with bounded maximum degree

Recall that $\mathcal{G}_\Delta$ is the family of graphs with maximum degree Δ. It is known that $\chi_{(1,0)}(\mathcal{G}_\Delta) \leq 2\Delta^2 2^\Delta$ [3]. Here we prove that $\chi_{(m,n)}(\mathcal{G}_\Delta) \leq 2(\Delta-1)^p.p^{(\Delta-1)} + 2$ for all $p = 2m+n \geq 2$ and $\Delta \geq 5$. Our result, restricted to the case $(m,n) = (1,0)$, slightly improves the upper bound of Kostochka, Sopena and Zhu [3].

Theorem 4.1. *For the family $\mathcal{G}_\Delta$ of graphs with maximum degree Δ we have $p^{\Delta/2} \leq \chi_{(m,n)}(\mathcal{G}_\Delta) \leq 2(\Delta-1)^p.p^{(\Delta-1)} + 2$ for all $p = 2m+n \geq 2$ and for all $\Delta \geq 5$.*

If every subgraph of a graph G have at least one vertex with degree at most d, then G is *d-degenerated*. Minimum such d is the *degeneracy* of G. To prove the above theorem we need the following result.

Theorem 4.2. *Let $\mathcal{G}'_\Delta$ be the family of graphs with maximum degree Δ and degeneracy $(\Delta-1)$. Then $\chi_{(m,n)}(\mathcal{G}'_\Delta) \leq 2(\Delta-1)^p.p^{(\Delta-1)}$ for all $p = 2m+n \geq 2$ and for all $\Delta \geq 5$.*

To prove the above theorem we need the following lemma.

Lemma 4.3. *There exists an (m,n)-colored complete mixed graph with property $Q^{t-1,j}_{1+(t-j)(t-2)}$ on $c = 2(t-1)^p.p^{(t-1)}$ vertices where $p = 2m+n \geq 2$ and $t \geq 5$.*

Proof. Let C be a random (m,n)-colored mixed graph with underlying complete graph. Let u, v be two vertices of C and the events $u \in N^a(v)$ for $a \in \{\pm\alpha_1, \pm\alpha_2, ..., \pm\alpha_m, \beta_1, \beta_2, ..., \beta_n\}$ are equiprobable and independent with probability $\frac{1}{2m+n} = \frac{1}{p}$. We will show that the probability of C not having property $Q^{t-1,j}_{1+(t-j)(t-2)}$ is strictly less than 1 when $|C| = c = 2(t-1)^p.p^{(t-1)}$. Let $P(J, \vec{a})$ denote the probability of the event $|N^{\vec{a}}(J)| < 1 + (t-j)(t-2)$ where J is a j-tuple of C and $\vec{a}$ is a j-vector for some $j \in \{0, 1, ..., t-1\}$. Call such an event a *bad event*. Thus,

$$
\begin{aligned}
P(J,\vec{a}) &= \sum_{i=0}^{(t-j)(t-2)} \binom{c-j}{i} p^{-ij}(1-p^{-j})^{c-i-j} \\
&< (1-p^{-j})^{c} \sum_{i=0}^{(t-j)(t-2)} \frac{c^i}{i!}(1-p^{-j})^{-i-j}p^{-ij} \\
&< 2e^{-cp^{-j}} \sum_{i=0}^{(t-j)(t-2)} c^i \\
&< e^{-cp^{-j}} c^{(t-j)(t-2)+1}.
\end{aligned} \tag{3}
$$

Let $P(B)$ denote the probability of the occurrence of at least one bad event. To prove this lemma it is enough to show that $P(B) < 1$. Let T^j denote the set of all j-tuples and W^j denote the set of all j-vectors. Then

$$
\begin{aligned}
P(B) = \sum_{j=0}^{t-1} \sum_{J\in T^j} \sum_{\vec{a}\in W^j} P(J,\vec{a}) &< \sum_{j=0}^{t-1} \binom{c}{j} p^j e^{-cp^{-j}} c^{(t-j)(t-2)+1} \\
&< \sum_{j=0}^{t-1} \frac{c^j}{j!} p^j e^{-cp^{-j}} c^{(t-j)(t-2)+1} \\
&= 2\sum_{j=0}^{t-1} \frac{p^j}{2^j} \frac{2^{j-1}}{j!} c^j e^{-cp^{-j}} c^{(t-j)(t-2)+1} \\
&< 2\sum_{j=0}^{t-1} \frac{p^j}{2^j} e^{-cp^{-j}} c^{(t-j)(t-2)+1+j}.
\end{aligned} \tag{4}
$$

Consider the function $f(j) = 2(p/2)^j e^{-cp^{-j}} c^{(t-j)(t-2)+1+j}$. Observe that $f(j)$ is the j^{th} summand of the last sum from equation (4). Now

$$
\begin{aligned}
\frac{f(j+1)}{f(j)} &= \frac{p}{2} \frac{e^{(p-1)cp^{-j-1}}}{c^{t-3}} \\
&> \frac{p}{2} \frac{e^{(p-1)cp^{-(t-1)}}}{c^{t-3}} \\
&> \frac{p}{2} \left(\frac{e^{2(p-1)(t-1)^{p-1}}}{c} \right)^{t-3}
\end{aligned} \tag{5}
$$

As $\frac{p-1}{p} > \frac{1}{2}$,

$$
\frac{(k-1)^{p-1}}{2} > \ln(k-1) \implies (p-1)(k-1)^{p-1} > \ln(k-1)^p.
$$

Furthermore,

$$
\frac{(p-1)}{\ln p}(k-1)^{p-1} > \frac{\ln 2}{\ln p} + (k-1) \implies (p-1)(k-1)^{p-1} > \ln(2p^{k-1}).
$$

Adding the above two inequalities we get

$$e^{2(p-1)(t-1)^{p-1}} > 2(t-1)^p p^{t-1} = c.$$

Hence $\frac{f(j+1)}{f(j)} > \frac{p}{2}$. Thus, using inequality (4) we get $P(B) < \sum_{j=0}^{t-1} f(j)$. This implies

$$P(B) < \begin{cases} \frac{(p/2)^t - 1}{(p/2)-1} f(0), & \text{if } p > 2 \\ tf(0), & \text{if } p = 2 \end{cases}$$

Case.1: $p > 2$.

$$\begin{aligned} P(B) &< 2.\frac{(p/2)^t - 1}{(p/2) - 1}.\frac{c^{(t-1)^2}}{e^{2(t-1)^p p^{t-1}}} \\ &< 4.\frac{(p/2)^t - 1}{p-2}.\left(\frac{c}{e^{2p^{t-1}}}\right)^{(t-1)^p} \\ &< 4.(p/2)^t.\left(\frac{c}{e^{2p^{t-1}}}\right)^{(t-1)^p} \\ &< \left(\frac{pc}{e^{2p^{t-1}}}\right)^{(t-1)^p} \end{aligned} \tag{6}$$

Now, we observe that

$$\begin{aligned} \ln(pc) &< \ln p + \ln 2 + p\ln(t-1) + (t-1)\ln p \\ &= t\ln p + p\ln(t-1) + \ln 2 \\ &< tp + p(t-1) + 2 \\ &< 2tp < 2p^{t-1} \end{aligned}$$

So from the inequality (6), we can say that $P(B) < 1$ for $p > 2$.

Case.2: $p = 2$.

$$\begin{aligned} P(B) &< 2t.\frac{c^{(t-1)^2}}{e^{(t-1)^2 2^t}} \\ &= 2t.\left(\frac{c}{e^{2^t}}\right)^{(t-1)^2} \\ &< \left(\frac{2tc}{e^{2^t}}\right)^{(t-1)^2} \end{aligned} \tag{7}$$

Observe that, $\ln c = 2\ln(t-1) + t\ln 2 < 2(t-1) + 2t = 4t - 2$.
Now, we see that

$$\ln(2tc) < 4t - 2 + 2t < 6t < 2^t \implies 2tc < e^{2^t} \implies \frac{2tc}{e^{2^t}} < 1$$

So from the inequality (7), we can say that $P(B) < 1$ for $p = 2$. $\square$

Now we are ready to prove Theorem 4.2.

Proof of Theorem 4.2. Suppose that G is an (m,n)-colored mixed graph with maximum degree Δ and degeneracy $(\Delta-1)$. By Lemma 4.3 we know that there exists an (m,n)-colored mixed graph C with property $Q^{\Delta-1,j}_{1+(\Delta-j)(\Delta-2)}$ on $2(\Delta-1)^p.p^{(\Delta-1)}$ vertices where $p=2m+n\geq 2$ and $\Delta \geq 5$. We will show that G admits a homomorphism to C.

As G has degeneracy $(\Delta-1)$, we can provide an ordering $v_1, v_2, ..., v_k$ of the vertices of G in such a way that each vertex v_j has at most $(\Delta-1)$ neighbors with lower indices. Let G_l be the (m,n)-colored mixed graph induced by the vertices $v_1, v_2, ..., v_l$ from G for $l \in \{1,2,...,k\}$. Now we will recursively construct a homomorphism $f: G \to C$ with the following properties:

- (i) The partial mapping $f(v_1), f(v_2), ..., f(v_l)$ is a homomorphism of G_l to C for all $l \in \{1,2,...,k\}$.
- (ii) For each $i > l$, all the neighbors of v_i with indices less than or equal to l has different images with respect to the mapping f.

Note that the base case is trivial, that is, any partial mapping $f(v_1)$ is enough. Suppose that the function f satisfies the above properties for all $j \leq t$ where $t \in \{1,2,...,k-1\}$ is fixed. Now assume that v_{t+1} has s neighbors with indices greater than $t+1$. Then v_{t+1} has at most $(\Delta-s)$ neighbors with indices less than $t+1$. Let A be the set of neighbors of v_{t+1} with indices greater than $t+1$. Let B be the set of vertices with indices at most t and with at least one neighbor in A. Note that as each vertex of A is a neighbor of v_{t+1} and has at most $\Delta-1$ neighbors with lesser indices, $|B| = (\Delta-2)|A| = s(\Delta-2)$. Let D be the set of possible options for $f(v_{t+1})$ such that the partial mapping is a homomorphism of G_{t+1} to C. As C has property $Q^{\Delta-1,j}_{1+(\Delta-j)(\Delta-2)}$ we have $|C| \geq 1+s(\Delta-1)$. So the set $D \setminus B$ is non-empty. Thus, choose any vertex from $D \setminus B$ as the image $f(v_{t+1})$. Note that this partial mapping satisfies the required conditions. $\square$

Finally, we are ready to prove Theorem 4.1.

Proof of Theorem 4.1. First we will prove the lower bound. Let G_t be a Δ regular graph on t vertices. Thus, G_t has $\frac{t\Delta}{2}$ edges. Then we have

$$k_t = \chi_{(m,n)}(G_t) \geq \frac{p^{\Delta/2}}{p^{\binom{k_t}{2}/t}}$$

using inequality (1) (see Section 3). If $\chi_{(m,n)}(G_t) \geq p^{\Delta/2}$ for some t, then we are done. Otherwise, $\chi_{(m,n)}(G_t) = k_t$ is bounded. In that case, if t is sufficiently large, then $\chi_{(m,n)}(G_t) \geq p^{\Delta/2}$ as $\chi_{(m,n)}(G_t)$ is a positive integer.

Let $G = (V, A \cup E)$ be a connected (m,n)-colored mixed graph with maximum degree $\Delta \geq 5$ and $p = 2m+n \geq 2$. If G has a vertex of degree at most $(\Delta-1)$ then it has degeneracy at most $(\Delta-1)$. In that case by Theorem 4.1 we are done.

Otherwise, G is Δ regular. In that case, remove an edge uv of G to obtain the graph G'. Note that G' has maximum degree at most Δ and has degeneracy at most $(\Delta-1)$. Therefore, by Theorem 4.1 there exists an (m,n)-colored complete mixed graph C on $2(\Delta-1)^p.p^{(\Delta-1)}$ vertices to which G' admits a f homomorphism to. Let G'' be the graph obtained by deleting the vertices u and v of G'. Note that the homomorphism f restricted to G'' is a homomorphism f_{res} of G'' to C. Now include two new vertices u' and v' to C and obtain a new graph C'. Color the edges or arcs between the vertices of C and $\{u',v'\}$ in such a way so that we can extend the homomorphism f_{res} to a homomorphism f_{ext} of G to C' where $f_{ext}(u) = u'$, $f_{ext}(v) = v'$ and

$f_{ext}(x) = f_{res}(x)$ for all $x \in V(G) \setminus \{u, v\}$. It is easy to note that the above mentioned process is possible.

Thus, every connected (m, n)-colored mixed graph with maximum degree Δ admits a homomorphism to C'. $\square$

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